

Polar Interpolants for Thin-Shell Microstructure Homogenization - Supplementary Document

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CCS CONCEPTS

• **Computer systems organization** → **Embedded systems**; *Redundancy*; Robotics; • **Networks** → Network reliability.

KEYWORDS

Wireless sensor networks, media access control, multi-channel, radio interference, time synchronization

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1 SUMMARY OF THE FULL ENERGY MODEL

1.1 Eigen-Strain

Recall that our energy model takes as input a 6D eigen-strain $\hat{x} = (\lambda_E, \mu_E, \theta_E, \lambda_\Lambda, \mu_\Lambda, \theta_\Lambda)$, where $(\lambda_E, \mu_E, \theta_E)$ is the eigen-representation of Green strain in 2D, and $(\lambda_\Lambda, \mu_\Lambda, \theta_\Lambda)$ is the eigen-representation of the shape operator.

For a full thin-shell eigen-strain $\hat{x}$, we consider two evaluations of the cylindrical-bending eigen-strain $\bar{x}$:

$$\bar{x}_\lambda(\hat{x}) = (\lambda_E, \mu_E, \theta_E, \lambda_\Lambda, \theta_\Lambda), \quad (1)$$

$$\bar{x}_\mu(\hat{x}) = (\lambda_E, \mu_E, \theta_E, \mu_\Lambda, \theta_\Lambda + \pi/2). \quad (2)$$

And we obtain the normalized strain x by normalizing principal strains based on the range of training data:

$$\begin{aligned} x(\bar{x}) &= (x^1 \dots x^5) \\ &= (\text{normalize}(\lambda_E), \text{normalize}(\mu_E), \theta_E, \text{normalize}(\lambda_\Lambda^*), \theta_\Lambda^*). \end{aligned} \quad (3)$$

1.2 RBF Interpolants

Each RBF interpolant is defined as:

$$\psi_i(x) = w^T \nabla u u \phi(r), \quad (4)$$

with $r = \|u\|$, u the polar difference vector, and ∇u its gradient, as detailed in the main paper. With RBF weights $w = (w^1 \dots w^5)$

and RBF center $x_i = (x_i^1 \dots x_i^5)$, we rewrite the interpolant for convenience as:

$$\psi_i(x) = f(x) \phi(r), \quad (5)$$

$$\text{with } f(x) = \sum_{j \in \{1,2,4\}} w^j (x^j - x_i^j) + \sum_{j \in \{3,5\}} w^j / 2 \sin(2(x^j - x_i^j)).$$

1.3 Energy

The fitted energy model $\Psi(x)$ uses pairs of RBF interpolants $\psi_i(x_\lambda)$ and $\psi_i(x_\mu)$, after normalizing the two cylindrical-bending eigen-strains (1) and (2). This yields the following expression for the fitted energy:

$$\Psi(x) = \sum_i \psi_i(x_\lambda) + \psi_i(x_\mu). \quad (6)$$

Finally, the evaluation of the runtime energy model Ψ_{full} uses four variants of the strain: the regular strain $\hat{x}$, a membrane-isotropic strain $\hat{x}_{\text{iso},E}(\hat{x}) = (\lambda_E, \lambda_E, 0, \lambda_\Lambda, \mu_\Lambda, \theta_\Lambda)$, a bending-isotropic strain $\hat{x}_{\text{iso},\Lambda}(\hat{x}) = (\lambda_E, \mu_E, \theta_E, \lambda_\Lambda, \lambda_\Lambda, 0)$, and a full isotropic strain $\hat{x}_{\text{iso}}(\hat{x}) = (\lambda_E, \lambda_E, 0, \lambda_\Lambda, \lambda_\Lambda, 0)$. These strains are then normalized as in (3) for their both cylindrical-bending versions. The expression for the runtime energy is then:

$$\begin{aligned} \Psi_{\text{full}} &= \alpha(\hat{x}) \Psi(x) + \alpha_{\text{iso},E}(\hat{x}) \Psi(x_{\text{iso},E}) \\ &\quad + \alpha_{\text{iso},\Lambda}(\hat{x}) \Psi(x_{\text{iso},\Lambda}) + \alpha_{\text{iso}}(\hat{x}) \Psi(x_{\text{iso}}), \end{aligned} \quad (7)$$

with $\omega(z) = e^{-z^2}$, $z_E = \lambda_E - \mu_E$, $z_\Lambda = \lambda_\Lambda - \mu_\Lambda$ and

$$\alpha(\hat{x}) = (1 - \omega(z_E))(1 - \omega(z_\Lambda)), \quad (8)$$

$$\alpha_{\text{iso},E}(\hat{x}) = \omega(z_E)(1 - \omega(z_\Lambda)), \quad (9)$$

$$\alpha_{\text{iso},\Lambda}(\hat{x}) = (1 - \omega(z_E))\omega(z_\Lambda), \quad (10)$$

$$\alpha_{\text{iso}}(\hat{x}) = \omega(z_E)\omega(z_\Lambda). \quad (11)$$

2 ENERGY FITTING ALGORITHM

The degrees of freedom of the energy fitting are the RBF centers $\{x_i\}$, the RBF weights $\{w_i\}$, the RBF radius (standard deviation) σ (common for all interpolants), and of course the number of interpolants P .

As mentioned in the main paper, we gather the training data in batches, corresponding to all training samples with the same principal stretch λ_E . The training samples are characterized by their normalized eigen-strain $x_{m,n}$ and their continuum stress $\frac{\partial \Psi}{\partial x} m, n$ (Please refer to the main paper for the computation of this stress). Index m runs over batches M , and index n runs over training samples N_m within a batch.

For each batch, we identify the maximum and minimum of each stress component, $\frac{\partial \Psi}{\partial x^j} m, \max$ and $\frac{\partial \Psi}{\partial x^j} m, \min$. Then, the operation $\text{diag}(s_m) \frac{\partial \Psi}{\partial x} m, n + o_m$ performs a per-batch stress normalization,

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with $s_m^j = 2/(\frac{\partial \Psi}{\partial x^j} m_{\max} - \frac{\partial \Psi}{\partial x^j} m_{\min})$ and $o_m^j = -(\frac{\partial \Psi}{\partial x^j} m_{\max} + \frac{\partial \Psi}{\partial x^j} m_{\min})/(\frac{\partial \Psi}{\partial x^j} m_{\max} - \frac{\partial \Psi}{\partial x^j} m_{\min})$. In practice, the offset o_m is irrelevant as it is applied to both the training samples and to the energy candidates, hence it cancels out in the error evaluation.

We define the fitting error as follows:

$$\varepsilon(\{w_i\}, \sigma) = \sum_{m \in M, n \in N_m} \left| s_m \left(\frac{\partial \Psi}{\partial x} (x_{m,n}, \{w_i\}, \sigma) - \frac{\partial \Psi}{\partial x} m_{n,n} \right) \right|^2. \quad (12)$$

The fitting algorithm iterates the following steps until the error reaches 10% or it stagnates:

- (1) Add an RBF ($P \leftarrow P + 1$) at the training strain sample $x_{m,n}$ furthest from the current RBFs. Distance is measured using the polar difference vector, to account for domain periodicity.
- (2) Run COBYLA to find the RBF weights and radius that minimize the error (12). We pass the radius σ as degree of freedom to COBYLA, and in the error evaluation we minimize (12) with respect to the RBF weights. This amounts to solving a linear system, as the error is quadratic in the weights.

3 DERIVATIVES OF THE RUNTIME MODEL

Our triangle-centered eigen-strain is a function of degrees of freedom q , which for the purpose of this document can be restricted to the vertex positions of the triangle and its flap neighbors. The runtime energy model $\Psi_{\text{full}}(\hat{x}(q))$ can be regarded as the composition of multiple functions, and we obtain its gradient and Hessian by chain-rule of these functions. In our implementation, we use analytical derivatives for $\Psi_{\text{full}}(\hat{x})$ and symbolic differentiation for $\hat{x}(q)$. Here we provide the analytical derivatives, and we refer to the code for the symbolic derivatives.

3.1 Derivatives of the Runtime Energy (7)

$$\begin{aligned} \frac{\partial \Psi_{\text{full}}}{\partial \hat{x}} &= \alpha(\hat{x}) \frac{\partial \Psi(x)}{\partial \hat{x}} + \alpha_{\text{iso},E}(\hat{x}) \frac{\partial \Psi(x_{\text{iso},E})}{\partial \hat{x}} \\ &+ \alpha_{\text{iso},\Lambda}(\hat{x}) \frac{\partial \Psi(x_{\text{iso},\Lambda})}{\partial \hat{x}} + \alpha_{\text{iso}}(\hat{x}) \frac{\partial \Psi(x_{\text{iso}})}{\partial \hat{x}} \\ &+ \Psi(x) \frac{\partial \alpha(\hat{x})}{\partial \hat{x}} + \Psi(x_{\text{iso},E}) \frac{\partial \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}} \\ &+ \Psi(x_{\text{iso},\Lambda}) \frac{\partial \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}} + \Psi(x_{\text{iso}}) \frac{\partial \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}}. \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial^2 \Psi_{\text{full}}}{\partial \hat{x}^2} &= \alpha(\hat{x}) \frac{\partial^2 \Psi(x)}{\partial \hat{x}^2} + \alpha_{\text{iso},E}(\hat{x}) \frac{\partial^2 \Psi(x_{\text{iso},E})}{\partial \hat{x}^2} \\ &+ \alpha_{\text{iso},\Lambda}(\hat{x}) \frac{\partial^2 \Psi(x_{\text{iso},\Lambda})}{\partial \hat{x}^2} + \alpha_{\text{iso}}(\hat{x}) \frac{\partial^2 \Psi(x_{\text{iso}})}{\partial \hat{x}^2} \\ &+ \Psi(x) \frac{\partial^2 \alpha(\hat{x})}{\partial \hat{x}^2} + \Psi(x_{\text{iso},E}) \frac{\partial^2 \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}^2} \\ &+ \Psi(x_{\text{iso},\Lambda}) \frac{\partial^2 \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}^2} + \Psi(x_{\text{iso}}) \frac{\partial^2 \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}^2} \\ &+ \frac{\partial \Psi(x)}{\partial \hat{x}} \frac{\partial \alpha(\hat{x})}{\partial \hat{x}} + \frac{\partial \Psi(x_{\text{iso},E})}{\partial \hat{x}} \frac{\partial \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}} \\ &+ \frac{\partial \Psi(x_{\text{iso},\Lambda})}{\partial \hat{x}} \frac{\partial \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}} + \frac{\partial \Psi(x_{\text{iso}})}{\partial \hat{x}} \frac{\partial \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}} \\ &+ \frac{\partial \alpha(\hat{x})}{\partial \hat{x}} \frac{\partial \Psi(x)}{\partial \hat{x}} + \frac{\partial \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}} \frac{\partial \Psi(x_{\text{iso},E})}{\partial \hat{x}} \\ &+ \frac{\partial \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}} \frac{\partial \Psi(x_{\text{iso},\Lambda})}{\partial \hat{x}} + \frac{\partial \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}} \frac{\partial \Psi(x_{\text{iso}})}{\partial \hat{x}}. \end{aligned} \quad (14)$$

3.2 Derivatives of the Blending Weights (8) - (11)

$$\omega'(z) = -\gamma z^{\gamma-1} e^{-z^\gamma}, \quad (15)$$

$$\omega''(z) = -\gamma(\gamma-1) z^{\gamma-2} e^{-z^\gamma} + \gamma^2 z^{2(\gamma-1)} e^{-z^\gamma}. \quad (16)$$

$$\frac{\partial z_E}{\partial \hat{x}} = (1, -1, 0, 0, 0, 0), \quad (17)$$

$$\frac{\partial z_\Lambda}{\partial \hat{x}} = (0, 0, 0, 1, -1, 0). \quad (18)$$

$$\frac{\partial \alpha(\hat{x})}{\partial \hat{x}} = -(1 - \omega(z_E)) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}} - (1 - \omega(z_\Lambda)) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}, \quad (19)$$

$$\frac{\partial \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}} = -\omega(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}} + (1 - \omega(z_\Lambda)) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}, \quad (20)$$

$$\frac{\partial \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}} = (1 - \omega(z_E)) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}} - \omega(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}, \quad (21)$$

$$\frac{\partial \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}} = \omega(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}} + \omega(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}. \quad (22)$$

$$\begin{aligned} \frac{\partial^2 \alpha(\hat{x})}{\partial \hat{x}^2} = & -(1 - \omega(z_E)) \omega''(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}} \\ & + \omega'(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & - (1 - \omega(z_\Lambda)) \omega''(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & + \omega'(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}}, \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial^2 \alpha_{\text{iso},E}(\hat{x})}{\partial \hat{x}^2} = & -\omega(z_E) \omega''(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}} \\ & - \omega'(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & + (1 - \omega(z_\Lambda)) \omega''(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & - \omega'(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}}, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial^2 \alpha_{\text{iso},\Lambda}(\hat{x})}{\partial \hat{x}^2} = & (1 - \omega(z_E)) \omega''(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}} \\ & - \omega'(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & - \omega(z_\Lambda) \omega''(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & - \omega'(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}}, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial^2 \alpha_{\text{iso}}(\hat{x})}{\partial \hat{x}^2} = & \omega(z_E) \omega''(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}} \\ & + \omega'(z_E) \omega'(z_\Lambda) \frac{\partial z_\Lambda}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & + \omega(z_\Lambda) \omega''(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_E}{\partial \hat{x}} \\ & + \omega'(z_\Lambda) \omega'(z_E) \frac{\partial z_E}{\partial \hat{x}}^T \frac{\partial z_\Lambda}{\partial \hat{x}}. \end{aligned} \quad (26)$$

3.3 Derivatives of the Fitted Energy (6)

Derivative of the strain normalization, just a scale matrix:

$$A = \frac{\partial x}{\partial \bar{x}} = \text{diag} \left(\frac{2}{\lambda_E^{\max} - \lambda_E^{\min}}, \frac{2}{\mu_E^{\max} - \mu_E^{\min}}, 1, \frac{2}{\lambda_\Lambda^{\max} - \lambda_\Lambda^{\min}}, 1 \right). \quad (27)$$

Derivatives of the cylindrical-bending eigen-strains, just selection matrices:

$$B_\lambda = \frac{\partial \bar{x}_\lambda(\hat{x})}{\partial \hat{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (28)$$

$$B_\mu = \frac{\partial \bar{x}_\mu(\hat{x})}{\partial \hat{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

Derivatives of the isotropic strains, also selection matrices:

$$C_{\text{aniso}} = \frac{\partial \hat{x}}{\partial \hat{x}} = I, \quad (30)$$

$$C_{\text{iso},E} = \frac{\partial \hat{x}_{\text{iso},E}}{\partial \hat{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (31)$$

$$C_{\text{iso},\Lambda} = \frac{\partial \hat{x}_{\text{iso},\Lambda}}{\partial \hat{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (32)$$

$$C_{\text{iso}} = \frac{\partial \hat{x}_{\text{iso}}}{\partial \hat{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

$$\frac{\partial \Psi(x)}{\partial \hat{x}} = \sum_i \frac{\partial \psi_i(x_\lambda)}{\partial x_\lambda} A B_\lambda C + \frac{\partial \psi_i(x_\mu)}{\partial x_\mu} A B_\mu C. \quad (34)$$

$$\frac{\partial^2 \Psi(x)}{\partial \hat{x}^2} = \sum_i C^T B_\lambda^T A^T \frac{\partial^2 \psi_i(x_\lambda)}{\partial x_\lambda^2} + C^T B_\mu^T A^T \frac{\partial^2 \psi_i(x_\mu)}{\partial x_\mu^2}. \quad (35)$$

3.4 Derivatives of the RBF Interpolants (5)

$$\frac{\partial \psi_i(x)}{\partial x} = f(x) \frac{\phi'(r)}{r} g(x) + \phi(r) \frac{\partial f(x)}{\partial x}, \quad (36)$$

$$\text{with } g(x) = u^T \nabla u = \begin{cases} x^j - x_i^j, & \text{if } j \in 1, 2, 4, \\ 1/2 \sin(2(x^j - x_i^j)), & \text{if } j \in 3, 5. \end{cases}$$

$$\frac{\partial^2 \psi_i(x)}{\partial x^2} = \phi(r) \frac{\partial^2 f(x)}{\partial x^2} + f(x) \left(\frac{\phi''(r)}{r^2} - \frac{\phi'(r)}{r^3} \right) g(x)^T g(x) \quad (37)$$

$$+ \frac{\phi'(r)}{r} \left(g(x)^T \frac{\partial f(x)}{\partial x} + \frac{\partial f(x)}{\partial x}^T g(x) + f(x) \frac{\partial g(x)}{\partial x}^T \right).$$

$$\frac{\partial f(x)}{\partial x} = \begin{cases} w^j, & \text{if } j \in 1, 2, 4, \\ w^j \cos(2(x^j - x_i^j)), & \text{if } j \in 3, 5. \end{cases} \quad (38)$$

$$\text{diag} \left(\frac{\partial^2 f(x)}{\partial x^2} \right) = \begin{cases} 0, & \text{if } j \in 1, 2, 4, \\ -2 w^j \sin(2(x^j - x_i^j)), & \text{if } j \in 3, 5. \end{cases} \quad (39)$$

$$\text{diag} \left(\frac{\partial g(x)}{\partial x} \right) = \begin{cases} 1, & \text{if } j \in 1, 2, 4, \\ \cos(2(x^j - x_i^j)), & \text{if } j \in 3, 5. \end{cases} \quad (40)$$

With a Gaussian RBF $\phi(r) = e^{-\frac{r^2}{2\sigma^2}}$, its derivatives are:

$$\phi'(r) = -\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad (41)$$

$$\phi''(r) = \left(\frac{r^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-\frac{r^2}{2\sigma^2}} \quad (42)$$

4 CONTROLLED VALIDATION TESTS

Fig. 1 to Fig. 10 show the controlled validation tests for all 10 microstructures.

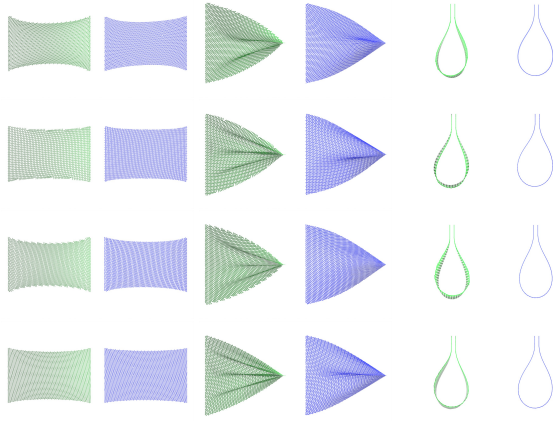


Figure 1: Microstr. 1 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

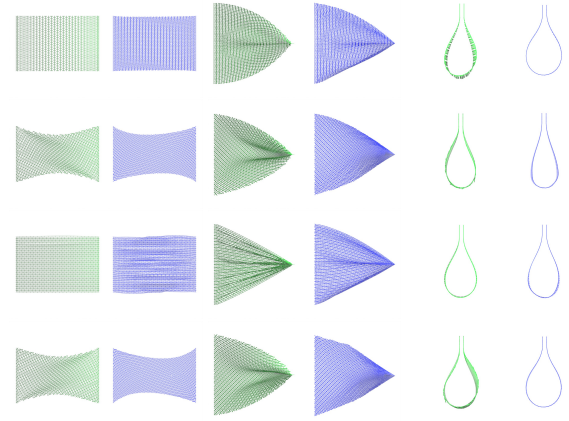


Figure 4: Microstr. 4 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

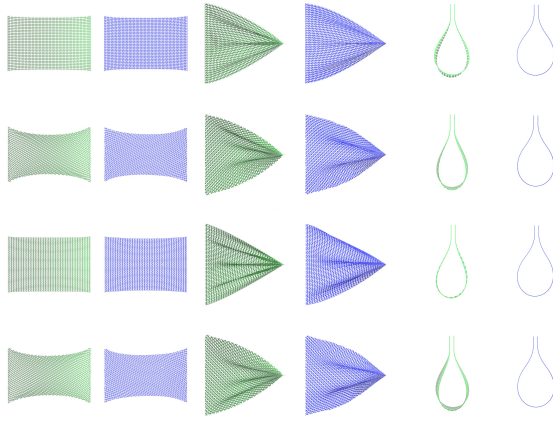


Figure 2: Microstr. 2 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

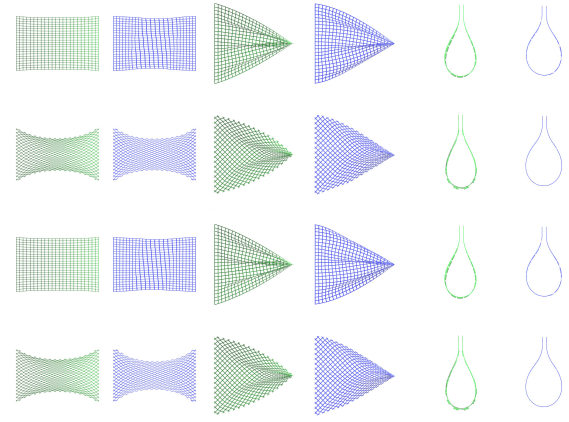


Figure 5: Microstr. 5 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

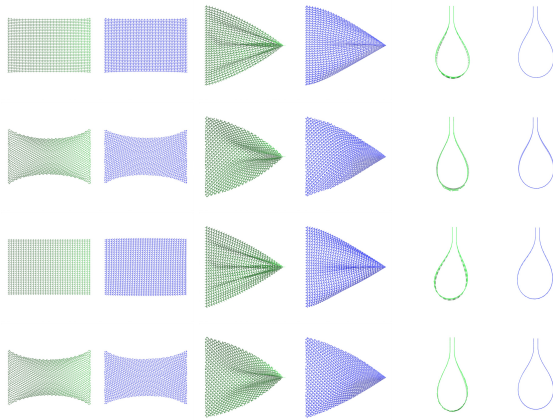


Figure 3: Microstr. 3 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

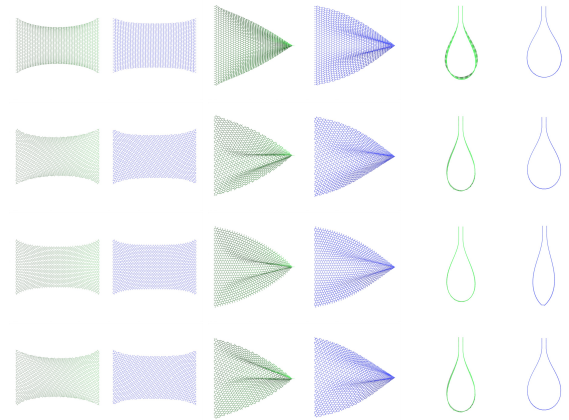


Figure 6: Microstr. 6 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

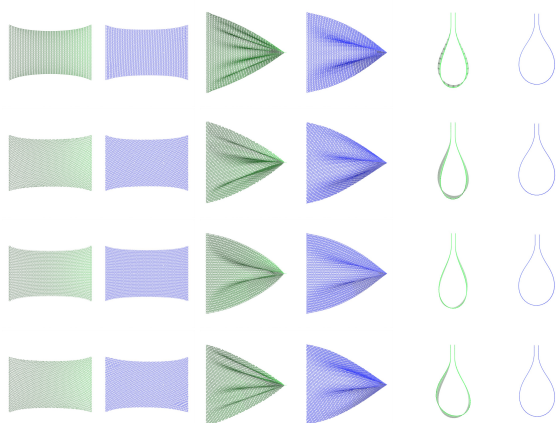


Figure 7: Microstr. 7 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

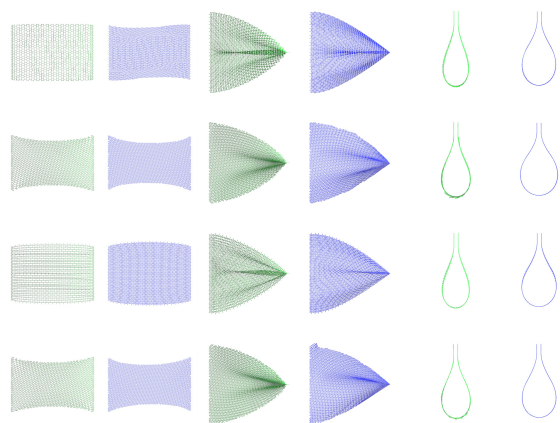


Figure 9: Microstr. 9 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

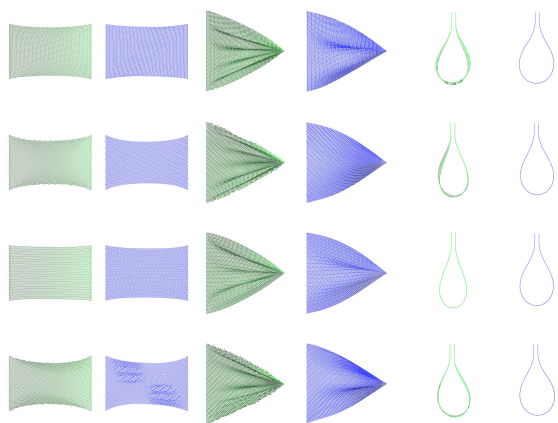


Figure 8: Microstr. 8 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)

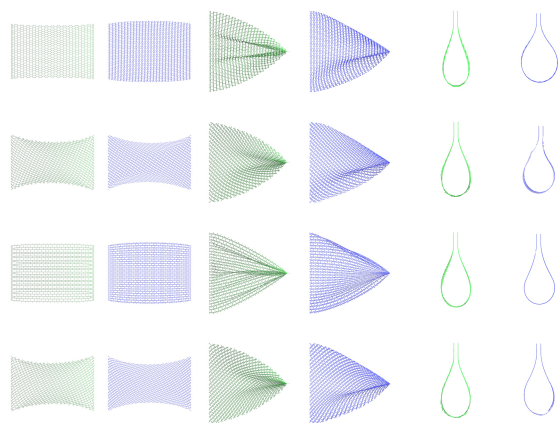


Figure 10: Microstr. 10 50% stretch, 90° twist, pear loop (0°, 45°, 90°, 135°)